Fuzzy Lattice Ordered Group

Vimala J

Abstract—In this paper, the definition of Fuzzy l- group with respect to Fuzzy partial order relation is introduced.   Some properties on Fuzzy l-groups is derived. The relation between Fuzzy I-group, DRL- group and Browerian Algebra are established..

Index Terms—Fuzzy lattice, Fuzzy partial order relation, Fuzzy I-group, Lattice ordered group

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1   INTRODUCTION

The concept of a fuzzy set was first introduced by Zadah [1] and this concept was adapted by Goguen [2] for fuzzy relations. Ajmal and Thomas [3] defined a fuzzy lattice as a fuzzy algebra. The theory of distributive lattices, being the major part in the present lattice theory, many mathematicians have introduced various types of elements satisfying certain distributive equations. Ore, O., has introduced and developed the concept of distributive elements in lattices, Gratzer, G., and Schmidt, E.T., have introduced and characterized standard elements in lattices and Birkhoff, G., has introduced neutral elements in lattices.   Later, Birkhoff,G., Gratzer, G., Schmidt E.T., Hashimoto, J., Kinugawa, S., and Iqbalunnisa have developed many equivalent conditions for an element of a lattice to be neutral.

In continuation of all the above, we define fuzzy lattice ordered group and develop some properties of fuzzy lattice ordered group.

2 PRELIMINARIES

2.1 DEFINITION

A non-empty set G is called a commutative lattice ordered group iff
1. $(G,+)$ is a commutative group
2. $(G,\le)$ is a lattice

$x \le y \Rightarrow a + x \le a + y$ for all a, x, y in G.

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2.2 Definition

A non-empty set G is called a commutative lattice ordered group iff
1. $(G,+)$ is a commutative group
2. $(G,\vee,\wedge)$ is a lattice
3. $a + (x \vee y) = (a + x) \vee (a + y)$

$a + (x \wedge y) = (a + x) \wedge (a + y)$ for all a, x, y in G.

2.3 Definition

Let X be a set. A function $A : X \times X \to [0,1]$ is called a fuzzy relation in X.

The fuzzy relation A in X is reflexive iff $A(x,x) = 1$ for all $x \in X$, A is transitive iff $A(x,z) \ge \sup \min(A(x,y), A(y,z))$, and A is antisymmetric iff $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$. A fuzzy relation A is a fuzzy partial order relation if A is reflexive, antisymmetric and transitive.

2.4 Definition

Let (X, A) be a fuzzy poset and let $B \subseteq X$. An element $u \in X$ is said to be an upper bound for a subset B iff $A(b,u) > 0$ for all $b \in B$. An upper bound u_0 for B is the least upper bound of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in X$ is said to be a lower bound for a subset B iff $A(v,b) > 0$ for all $b \in B$. A lower bound v_0 for B is the greatest lower bound of B iff $A(v, v_0) > 0$ for every lower bound v for B.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

2.5 Definition

Let (X, A) be a fuzzy poset. (X, A) is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

2.6 Proposition

Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then

1. $A(x, x \vee y) > 0$, $A(y, x \vee y) > 0$, $A(x \wedge y, x) > 0$, $A(x \wedge y, y) > 0$.
2. $A(x, z) > 0$ and $A(y, z) > 0$ implies $A(x \vee y, z) > 0$.
3. $A(z, x) > 0$ and $A(z, y) > 0$ implies $A(z, x \wedge y) > 0$.
4. $A(x, y) > 0$ iff $x \vee y = y$
5. $A(x, y) > 0$ iff $x \wedge y = x$
6. If $A(y, z) > 0$, then $A(x \wedge y, x \wedge z) > 0$ and $A(x \vee y, x \vee z) > 0$.

2.7 Definition

A system $A = \{A, +, \leq\}$ is called a dually residuated lattice ordered group or DRL – group if

(i) $(A, +)$ is an abelian group

(ii) $(A, \leq)$ is a lattice

(iii) $b \leq c \Rightarrow a + b \leq a + c$ for all a, b, c in A

(iv) Given a, b in A there exist a least element $x = a - b$ in A such that $b + x \geq a$

2.8 Definition

A non-empty set B is called a Browerian Algebra if and only if

(i) $(B, \leq)$ is a lattice
(ii) B has a least element
(iii) To each a, b in B, there exist a least $x = a - b$ in B such that $b \vee x \geq a$

3 FUZZY LATTICE ORDERED GROUP

3.1 Definition

A non-empty set G is called a fuzzy lattice ordered group iff

(i) $(G, +)$ is a group.
(ii) (G, R) is a fuzzy lattice.
(iii) $R(x, y) \geq 0 \Rightarrow R(a + x, a + y) \geq 0$ for all $x, y, a \in G$.
Here $R : G \times G \to \}[0,1]$ is a fuzzy partial order relation.

3.2 Definition

A non-empty set G is called a fuzzy lattice ordered group iff

(i) $(G, +)$ is a group.
(ii) $(G, \vee, \wedge)$ is a fuzzy lattice.
(iii) $R(a + (x \vee y), (a + x) \vee (a + y)) = 1$ for all $a, x, y \in G$.
Here $R : G \times G \to [0,1]$ is a fuzzy partial order relation.

Example

$(\mathfrak{R}, +, A)$ is a fuzzy lattice ordered group.

Theorem 1

Two definitions for fuzzy lattice ordered group are equivalent.

Proof

It is enough to prove that the following two conditions are equivalent.

$R(x, y) \geq 0 \Rightarrow R(a + x, a + y) \geq 0$ (1)

$R(a + (x \vee y), (a + X) \vee (a + y)) = 1$ (2)

$(1) \Rightarrow (2)$

Assume that $R(x, y) \geq 0 \Rightarrow R(a + x, a + y) \geq 0$

$R(a + (x \vee y), (a + x) \vee (a + y))$

$= R(a + y, a + y)$

$= 1$ (since R is reflexive)

$(2) \Rightarrow (1)$

Suppose $R(x, y) \geq 0$

Now $R(a + (x \vee y), (a + x) \vee (a + y)) = 1$

$\Rightarrow a + (x \vee y) = (a + x) \vee (a + y)$

$\Rightarrow a + y = (a + x) \vee (a + y)$

$\Rightarrow a + x \leq a + y$

$\Rightarrow R(a + x, a + y) \geq 0$

Here the two definitions are equivalent.

Theorem 2

Any fuzzy lattice ordered group is a DRL-group.

Proof

Let $(G, +, R)$ be a fuzzy lattice ordered group. Here R is a fuzzy partial order relation.

It is enough to prove that

"if $a, b \in G$, there exist a least element x in G such that $R(a, b + x) > 0$".

Let $a, b \in G \Rightarrow a, -b \in G$

$\Rightarrow a + (-b) \in G$

$\Rightarrow a - b \in G$

$\Rightarrow$ There exist a least $x = a - b$ in G such that

$R(a, b + x) = R(a, b + a - b)$

$= R(a, a) = 1 > 0$

$\Rightarrow R(a, b + x) > 0$

4 SOME PROPERTIES OF FUZZY LATTICE ORDERED GROUP

Property 1

$R\big[(a \vee b), \big[(a-b) \vee 0\big]+b\big]=1$ for all $a,b \in G$

Proof

Let $a,b \in G$ be arbitrary

$$\big[(a-b) \vee 0\big] = b + \big[(a-b) \vee 0\big]$$
$$= \big[b+(a-b)\big] \vee (b+0)$$
$$= a \vee b$$
$$\Rightarrow R(a \vee b),\big((a-b) \vee 0\big)+b = 1$$

Property 2

$R(a,b) \geq 0 \Rightarrow R(a-c,b-c) \geq 0$ and
$R(c-b,c-a) \geq 0$ for all a,b,c in G.

Proof

Let a,b,c in G be arbitrary.
Suppose $a \leq b$
$$\Rightarrow a \vee b = b \text{ and } a \wedge b = a$$
To prove $R(a-c,b-c) \geq 0$ and
$$R(c-b,c-a) \geq 0$$

Claim

$$(a-c) \vee (b-c) = b-c \text{ and}$$
$$(c-b) \vee (c-a) = c-a$$
Now $(a-c) \vee (b-c) = \big[(a-c)-(b-c) \vee 0\big]+(b-c)$ by prop. 1
$$= \big[(a-c-b+c) \vee 0\big]+b-c$$
$$= \big[(a-b) \vee 0\big]+(b-c)$$
$$= \big[(a-b) \vee 0 + b\big]-c$$
$$= (a \vee b)-c$$
$$= (b-c)$$
$$\Rightarrow (a-c) \vee (b-c) = b-c$$
Also $a \leq b \Rightarrow -b \leq -a$
$$\Rightarrow c+(-b) \leq c+(-a)$$
$$\Rightarrow c-b \leq c-a$$
$R(a-c,b-c) \geq 0$ and
$R(c-b,c-a) \geq 0$

Property 3

$R\big((a \vee b)-c,(a-c) \vee (b-c)\big)=1$ for all $a,b,c \in G$.

Proof

Let $a,b,c \in G$ be arbitrary.
$$\Rightarrow \big[(a-c) \vee (b-c)\big]+c$$
$$= c+\big[(a-c) \vee (b-c)\big]$$
$$= c+(a-c) \vee c+(b-c)$$
$$= a \vee b$$
$$\Rightarrow (a-c) \vee (b-c) = (a \vee b)-c$$
$$\Rightarrow R\big((a \vee b)-c,(a-c) \vee (b-c)\big)=1$$

Property 4

$$R\big(a-(b \vee c),(a-b) \wedge (a-c)\big)=1$$

Proof

Let $a,b,c \in G$ be arbitrary.
We have $b,c \leq b \vee c$
$$\Rightarrow -(b \vee c) \leq -b,-c$$
$$\Rightarrow a-(b \vee c) \leq a-b,a-c$$
$$\Rightarrow a-(b \vee c) \leq (a-b) \wedge (a-c)$$

Also $\big[(a-b) \wedge (a-c)\big]+(b \vee c)$
$$= \big[(a-b) \wedge (a-c)\big]+b \vee \big[(a-b) \wedge (a-c)\big]+c$$
$$\leq \big[(a-b)+b\big] \vee \big[(a-c)+c\big]$$
$$\leq a \vee a = a$$
$$\Rightarrow R\big(a-(b \vee c),(a-b) \wedge (a-c)\big)=1$$

Property 5

$R(b,a) \leq 0 \Rightarrow R\big((a-b)+b,a\big)=1$ for all $a,b \in G$.

Proof

Let a,b in G be arbitrary.
Assume that $R(b,a) \leq 0$
$$\Rightarrow a-b \geq 0$$
$$\Rightarrow (a-b) \vee 0 = a-b$$
By prop. (1) $\big[(a-b) \vee 0\big]+b = a \vee b$
$$\Rightarrow (a-b)+b = a \vee b$$
$$\Rightarrow (a-b)+b = a \text{ since } R(b,a) \leq 0$$
Thus $R(b,a) \leq 0 \Rightarrow R\big((a-b)+b,a\big)=1$

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